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SEPARATRIX CROSSING IN THE DYNAMICS OF A DUAL-SPIN SATELLITE†

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The evolution of the rotation of a gyrostat satellite with slow rotor spinup is described in the adiabatic approximation. Formulae are obtained for the probabilities of various outcomes of the evolution. Probability phenomena arise owing to separatrix crossing. © 2001 Elsevier Science Ltd. All rights reserved.

Dual-spin satellites constitute one of the main types of unmanned spacecraft. In the principal approximation, such a satellite consists of two rigid bodies connected by a rigid shaft. One of the bodies – the platform – has zero (small) angular velocity of rotation; the other – the rotor – is spinning rapidly relative to the platform. An onboard electric motor regulates the angular velocity of the rotor relative to the platform. The large angular momentum of the system stabilizes the satellite in an inertial frame of reference, while the non-rotating platform enables investigations to be conducted in a given space direction.

If the shaft is aligned with a principal central axis of inertia of one of the bodies, which is dynamically symmetrical about that axis, the system of two rigid bodies is a gyrostat. In the simplest case, the shaft is aligned with the common direction of the principal central axes of inertia of both bodies. Such a system is called an axial gyrostat. Possible complications of the problem arise in view of the fact that both bodies may have a triaxial ellipsoid of inertia, and in addition one or both bodies may be dynamically unbalanced relative to the shaft direction.

After being placed in orbit, the platform and the rotor are rigidly connected and rotate about the shaft direction as a single rigid body. Then the motor spins up the rotor relative to the platform, in a direction coinciding with that of the initial spin. Thus, the angular velocity of the platform tends to zero, while the angular momentum of the rotor becomes equal to the initial angular momentum of the system.

It is well known [1] that even in the case of an axial gyrostat the satellite may overturn in the course of the above process: the final rotation of the device will take place about a principal axis of inertia orthogonal to the direction of the initial rotation. The reason is that, during rotor spinup, the phase trajectories of the system, lying on the two-dimensional constant angular momentum surface, may cross an instantaneous separatrix of the unperturbed problem (a gyrostat with a constant relative angular velocity of rotation) and reach qualitatively different domains of final motion. If the rotor is spinning at a slowly varying angular velocity, small changes in the initial conditions of the problem will cause the system to fall into the different domains into which the separatrices divide the constant angular momentum surface, and one can therefore use a probability-theoretic approach to investigate the evolution of the rotation. Numerically determined probabilities have been published for different outcomes of the evolution of rotation in this problem [1].

In this paper, previous results [2] will be used to obtain analytical expressions for the probabilities of different outcomes of the evolution of rotation in an axial gyrostat in the case of slowly varying angular velocity of the rotor; computations using finite formulae will be compared with the results of numerical integration of the initial equations of the problem.

1. THE EQUATIONS OF MOTION

Following a previously developed approach [1], consider an axial gyrostat in which the platform is dynamically symmetrical about the shaft, while the rotor has a triaxial ellipsoid of inertia (the shaft being aligned with a principal central axis of inertia of the rotor). Let x_1, x_2 and x_3 denote the principal central

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axes of inertia of the gyrostat, where the x_1 axis is the axis of the rotor and the frame of reference is rigidly attached to the rotor.

The equations of motion of the system are

$$\frac{dh_1}{dt}=\frac{I_2-I_3}{I_2I_3}h_2h_3,\quad \frac{dh_2}{dt}=\left(\frac{I_3-I_R}{I_3I_R}h_1-\frac{h_a}{I_R}\right)h_3$$
$$\frac{dh_3}{dt}=\left(\frac{I_R-I_2}{I_2I_R}h_1+\frac{h_a}{I_R}\right)h_2,\quad \frac{dh_a}{dt}=g_a \tag{1.1}$$

where g_a is the torque applied by the rotor to the platform, h_a is the angular momentum of the platform about the x_1 axis, h_i is the angular momentum of the gyrostat relative to the x_1 axis, I_R is the moment of inertia of the rotor about the x_1 axis, and I_i are the principal central moments of inertia of the gyrostat ($i = 1, 2, 3$).

Since there are no external forces, the angular momentum of the system is conserved, so that

$$h^2=h_1^2+h_2^2+h_3^2=\text{const} \tag{1.2}$$

We will introduce the following change of variables [1]

$$x_i=h_i/h,\quad i=1,2,3$$
$$\mu=h_a/h,\quad \tau=ht/I_R,\quad \varepsilon=-g_aI_R/h^2$$

Derivatives with respect to dimensionless time τ are denoted by a dot, and we define dimensionless inertia parameters by

$$i_j=1-I_R/I_j,\quad j=1,2,3$$

Then the dimensionless equations of motion of the system become [1]

$$\dot{x}_1=(i_2-i_3)x_2x_3,\quad \dot{x}_2=(i_3x_1-\mu)x_3,\quad \dot{x}_3=(\mu-i_2x_1)x_2,\quad \dot{\mu}=-\varepsilon \tag{1.3}$$

and the dimensionless integral of the squared modulus of the angular momentum is

$$x_1^2+x_2^2+x_3^2=1 \tag{1.4}$$

We will henceforth assume that the angular acceleration of the rotor is small, $\varepsilon \ll 1$, and, to fix our ideas, that $i_1 > i_2 > i_3$, corresponding to a gyrostat with $I_1 > I_2 > I_3$.

Initially, the gyrostat is rotating as a rigid body; thus the initial conditions of the problem (x_1, x_2, x_3, μ) are such that $\mu = \mu_0 = x_1i_1$. If the gyrostat is initially rotating about an axis near the shaft axis, the initial data of the problem are close to $(1, 0, 0, i_1)$. During rotor spinup, the parameter μ, characterizing the angular velocity of rotation of the platform, decreases from μ_0 to zero.

The expression for the dimensionless kinetic energy of rotation of the gyrostat has the form

$$2T=(x_1-\mu)^2+(1-i_2)x_2^2+(1-i_3)x_3^2+(1-i_1)\mu^2/i_1$$

It is convenient [1] to introduce the following function together with T

$$y=2T-\mu^2/i_1-1=-2\mu x_1-i_2x_2^2-i_3x_3^2 \tag{1.5}$$

We have

$$\dot{y}=2\varepsilon x_1 \tag{1.6}$$

and in unperturbed motion ($\varepsilon = 0$) the quantity y is constant.

2. BIFURCATION OF THE EQUILIBRIA OF THE SYSTEM AND SLOW SEPARATRIX CROSSING

The phase trajectories of system (1.3), considered on the sphere (1.4) with μ = const, are level curves of the function y (1.5). Rebuildings of the phase portrait of the system are determined by bifurcations of the singular point (1, 0, 0).

Analysis of the right-hand sides of the equations shows that when $i_2 < \mu$ this point is a centre (Fig. 1). At $i_2 = \mu$ bifurcation occurs: the singular point becomes unstable and two stable singular points are formed, with coordinates

$$x_1 = \mu / i_2, \quad x_2 = \pm\sqrt{1-\mu^2/i_2^2}, \quad x_3 = 0 \tag{2.1}$$

When $i_3 < \mu < i_2$, there are two separatrices on the sphere (1.4), passing through the unstable singular point (1, 0, 0) and encircling the stable singular points (2.1). These separatrices divide the sphere into three domains G_1, G_2 and G_3 (Fig. 2). The area of the domain G_l will be denoted by $S_l = S_l(\mu)$, $S_1 = S_2$.

When $\mu = i_3$ a new bifurcation occurs: the point (1, 0, 0) again becomes stable and two unstable singular points are formed, with coordinates

$$x_1 = \mu / i_3, \quad x_2 = 0 \quad x_3 = \pm\sqrt{1-\mu^2/i_3^2} \tag{2.2}$$

Thus, when $0 < \mu < i_3$ there are four intersecting separatrices on the constant angular momentum sphere. They divide the sphere into four domains $G_1, \ldots, G_4$ (Fig. 3).

As $\mu \to 0$ the centres of the domains G_1 and G_2 (stable singular points (2.1)) tend to the points (0, ±1, 0), while the unstable singular points (2.2) tend to the points (0, 0, ±1).

We now consider the complete system (1.3) for small $\varepsilon > 0$. The divergence of the velocity vector of the motion in this system is equal to zero. Consequently, phase volume is conserved during the motion, and so phase volume (area) on the sphere (1.4) is also conserved. For small ε, therefore, system (1.3) has an adiabatic invariant [3]: the area S of the domain bounded by the instantaneous unperturbed trajectory (μ = const, y = const) on the sphere (1.4) passing through the phase point, considered in the principal approximation, remains unchanged when the phase point moves away from the separatrix (one can choose either of the two domains into which the unperturbed phase trajectory divides the sphere (1.4)). The adiabatic invariance of S may be used to describe the motion up to the time it reaches the separatrix [2].

Let the motion in system (1.3) begin at $\mu = \mu_0 \in (i_2, i_1)$, and suppose the initial unperturbed phase trajectory ($\mu = \mu_0$) bounds a domain of area S_0 on the sphere; to fix our ideas, we take the domain containing the point (1, 0, 0).

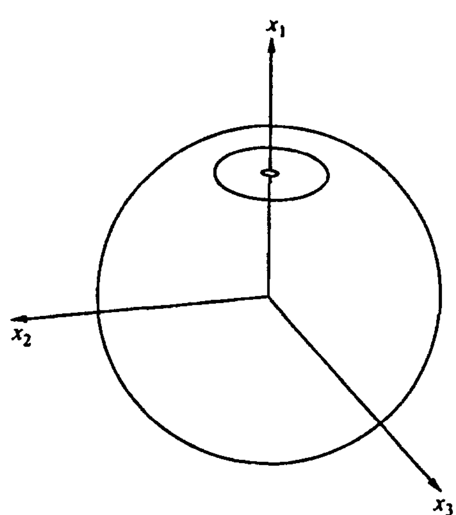

Fig. 1

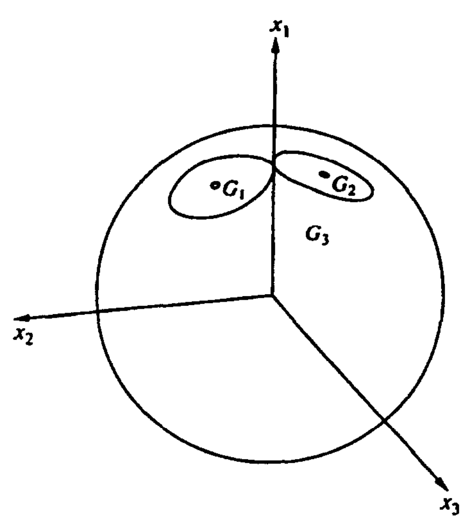

Fig. 2

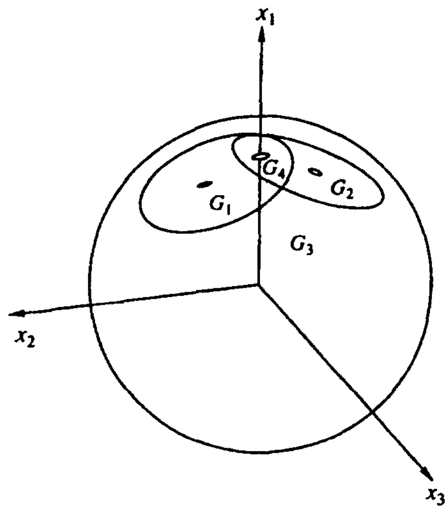

Fig. 3

If $S_0 < 2S_1(i_3) = \pi(1 - \sqrt{i_3/i_2})$, then, as shown by considering the adiabatic approximation, the perturbed phase trajectory crosses the separatrix at some $\mu \approx \mu_*$, where μ_* is a root of the equation $S_0 = 2S_1(\mu_*)$, $\mu_* \in (i_3, i_2)$. After crossing the separatrix, the trajectory is trapped in one of the domains G_1 or G_2; further motion occurs in such a way that the area bounded by the unperturbed trajectory (μ = const) in the domains G_1 and G_2 remains approximately constant and equal to $S_1(\mu_*)$. As $\mu \to 0$, the trajectory loops around the x_2 axis, so that the gyrostat overturns. The initial conditions (at $\mu = \mu_0$) corresponding to capture in S_1 or S_2 are mixed for small ε, so that it makes sense to speak of the probabilities of capture in the domain G_1 or G_2. It follows from the symmetry of the problem that these probabilities are equal to 1/2 (the definition of these probabilities will be discussed below).

Now let $S_0 > 2S_1(i_3)$. Then the perturbed phase trajectory will cross the separatrix at some $\mu \approx \mu_*$, where μ_* is a root of the equation $S_0 = S_4(\mu_*) + 2S_1(\mu_*)$, $\mu_* \in (i_2, 0)$. After crossing the separatrix, the phase trajectory will be trapped in one of the domains G_1, G_2, G_4, further motion will occur in such a way that the area bounded by the unperturbed phase trajectory will remain approximately constant, equal to the area at $\mu = \mu_*$ of the domain in which capture has taken place ($S_1(\mu_*)$ for capture in G_1 or G_2 and $S_4(\mu_*)$ for capture in G_4). Capture in G_1 and G_2 means that the gyrostat has overturned; capture in G_4 implies rotation analogous to the initial rotation. The initial conditions corresponding to capture in G_1, G_2 and G_4 are mixed for small ε, so that it makes sense to speak of the probabilities of capture in one of these domains.

A probability-theoretical approach to problems of this type was introduced in [4, 5]. The probability $P_l(M_0)$ of capture of a trajectory with initial point $M_0 = (x_{10}, x_{20}, x_{30})$ is defined as follows [5]. Let $U^{(\delta)}$ be a disk of radius δ with centre M_0 on the sphere (1.4) and let $U_l^{(\delta,\varepsilon)}$ be the set of initial data (at $\mu = \mu_0$) in $U^{(\delta)}$ to which trajectories with capture in G_l correspond. Then we define

$$P_l(M_0) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{\operatorname{mes} U_l^{(\delta,\varepsilon)}}{\operatorname{mes} U^{(\delta)}} \tag{2.3}$$

where mes(·) is area on the sphere. The limit (2.3) exists and may be calculated by the formula [2]

$$P_4(M_0) = \frac{dS_4/d\mu}{2dS_1/d\mu + dS_4/d\mu}, \quad P_{1,2}(M_0) = \frac{1}{2}(1 - P_4(M_0)) \tag{2.4}$$

In this formula one must evaluate the derivatives $dS_l/d\mu$ when $\mu = \mu_*$, where μ_* is the value of the parameter μ, computed in the adiabatic approximation, at which a phase trajectory with initial point M_0 will cross the separatrix.

The equations of the separatrix are expressed in terms of elementary functions using integral (1.5) of the unperturbed problem. Direct computation of the areas and their derivatives leads to the following result

$$S_{4,1}=\pi\left(1-\frac{\mu}{\sqrt{i_2 i_3}}\right)\pm 2I$$

$$I=\arcsin\frac{1}{\lambda}-\frac{\sqrt{p^2-\lambda^2}}{2p}\sum_{\pm}\arcsin\frac{p\pm\lambda^2}{\lambda(p\pm 1)}$$

$$\lambda^2=\frac{i_2 i_3-\mu^2}{i_3(i_2-i_3)},\quad p^2=\frac{i_2}{i_2-i_3}$$

$$\frac{\partial S_{4,1}}{\partial\mu}=-\frac{\pi}{\sqrt{i_2 i_3}}\pm 2\frac{\partial I}{\partial\mu} \qquad (2.5)$$

$$\frac{\partial I}{\partial\mu}=\frac{\mu}{i_3(i_2-i_3)}\left[\frac{1}{\lambda^2\sqrt{\lambda^2-1}}-\frac{1}{2p\sqrt{p^2-\lambda^2}}\sum_{\pm}\arcsin\frac{p\pm\lambda^2}{\lambda(p\pm 1)}-\right.$$

$$\left.-\frac{\sqrt{p^2-\lambda^2}}{2p\lambda^2}\sum_{\pm}\frac{p\mp\lambda^2}{\sqrt{\lambda^2(p\pm 1)^2-(p\pm\lambda^2)^2}}\right]$$

A series of computations were carried out to compare the probabilities of different outcomes of the evolution of rotation obtained using the above closed formulae and by numerical integration of Eqs (1.3). The domain of initial conditions for Eqs (1.3) for which the values of x_2 and x_3 lie inside the square $0 \leqslant x_2 \leqslant 0.05$; $0.5 \leqslant x_3 \leqslant 0.55$ was chosen on the unit sphere (1.4). Figure 4 illustrates one result of evolution of the gyrostat's rotation during rotor spinup, given different initial data for Eqs (1.3), situated on the unit sphere (1.4) and having values x_2 and x_3 in a fragment of the square just defined (the complete picture differs only slightly from that shown here). A final motion with initial conditions inside the hatched strips occurs in the domain G_4; the remaining field of the square consists of the initial data that lead to motion in the domains G_1 and G_2, that is, overturn of the satellite. The following values were chosen for the system parameters: $i_1 = 0.7$, $i_2 = 0.6$, $i_3 = 0.5$ and $\varepsilon = 10^{-4}$. As the results of numerical integration have shown, with these parameter values, the probabilities of a trajectory entering the domain G_4 and the union of the domains G_1 and G_2 are 0.18 and 0.82, respectively. Computations based on closed formulae (2.4) and (2.5) give 0.17 and 0.83. The value of μ corresponding to the separatrix crossing time was chosen as the mean value for all trajectories in the above square: $\mu = 0.4902$.

3. FAST SEPARATRIX CROSSING

We will now consider the case when the rotor spins up rapidly: $\varepsilon \gg 1$. Put $\nu = 1/\varepsilon \ll 1$. Spinup begins when $\mu = \mu_0$ and ends at $\mu = 0$. In the principal approximation with respect to ν, it may be assumed that the quantities x_i cannot be changed during spinup. The gyrostat will not overturn if at the initial time, when $\mu = \mu_0$, the phase trajectory is in the domain G_3 constructed for $\mu = 0$, that is, if at the

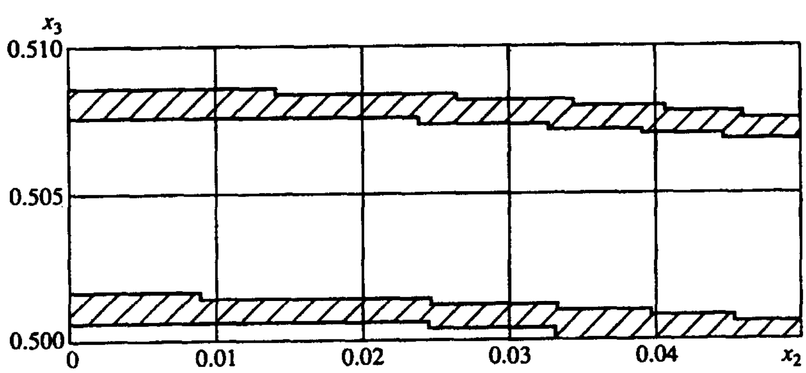

Fig. 4

initial time one has $i_2^2x_2^2 + i_3^2x_3^2 < i_2^2$. In the next approximation with respect to ν we find that during spinup the coordinates x_i vary by amounts Δx_i, where

$$\Delta x_2 = \nu(\mu_0 i_3 x_1 x_3 - \mu_0^2 x_3 / 2), \quad \Delta x_3 = \nu(\mu_0^2 x_2 / 2 - \mu_0 i_2 x_1 x_2) \tag{3.1}$$

Overturn does not occur if the initial point (x_1, x_2, x_3) (when $\mu = \mu_0$) is such that the point $(x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3)$ is situated in the domain G_3 constructed for $\mu = 0$. Use of relations (3.1) yields the following condition for non-overturn

$$i_2^2 x_2^2 + i_3^2 x_3^2 - \nu(i_2^2 - i_3^2)\mu_0^2 x_2 x_3 < i_2^2$$

In conclusion, we note that separatrix crossing is of crucial importance in many present-day problems of rigid body dynamics. A survey of problems of this kind may be found in [6].

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